An incomplete review of L^2-methods

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Contents

Plurisubharmonic functions

Definition

A function $\varphi : \Omega \to [-\infty, +\infty)$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic (psh for short), if

- ϕ is upper semi-continuous;
- $\phi|_{L\cap\Omega}$ is subharmonic for every complex line $L \subset \mathbb{C}^n$.

Example:

$$\phi = \log\Big(\sum_{1\le j\le N} |f_j|^2\Big)$$

where f_j are holomorphic functions on Ω.

Basic properties:

1. ϕ satisfies sub mean-value inequality and thus maximum principle.
2. $i\partial\bar{\partial}\phi \geq 0$ in the sense of currents. ϕ is called strictly psh, if $i\partial\bar{\partial}\phi > 0$ in the sense of currents.
3. ϕ_m psh, and decreasing to ϕ, then ϕ is also psh.
4. ϕ psh, then $\phi_\epsilon := \phi * \rho_\epsilon$ is psh on Ω_ϵ, and $\phi_\epsilon \searrow \phi$ on every Ω_ϵ.

Hörmander's L^2-estimate

In complex geometry, we have the following two powerful theorems related to psh functions.

Theorem (Hörmander's L^2-estimate, '65)

Let $\Omega \subset \mathbb{C}^n$ be a bounded smooth strictly pseudoconvex domain, and $\phi \in Psh(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$. Let f be a $(0,1)$-form with $\bar\partial f = 0$, then there is a function u on Ω, such that $\bar\partial u = f$, with the following estimate

$$\int_\Omega |u|^2 e^{-\phi} \le \int_\Omega |f|^2_{i\partial\bar\partial\phi} e^{-\phi},$$

provided that the latter is finite.

L. Hörmander, L^2-estimates and existence theorems for the $\bar\partial$ operator, Acta. Math. 113, 89–152, 1965.

Motivation of Hörmander's L^2-estimate

Answer a question of Bergman:

Theorem (Hörmander '65)

Let D be a pseudoconvex domain in $\mathbb{C}^n$ and ρ be the distance function to the boundary of D. Let z_0 be a point in ∂D such that ∂D is of $\mathcal{C}^2$-smooth at z_0 and the Levi form of ρ at z_0 in the plane

$$\sum \frac{\partial \rho}{\partial z_j}(z_0) t_j = 0$$

is positive definite. Let $k(z_0)$ be the product of the $n-1$-eigenvalues of this form. Then the Bergman kernel $K_D(z, w)$ of D satisfies

$$\rho(z)^{n+1} K_D(z, z) \to k(z_0) \frac{n!}{4\pi^n} \qquad as \quad z \to z_0.$$

Proof

- Use Hörmander's L^2-estimate to get a localization principle for Bergman kernels near strictly pseudoconvex boundary points, and then reduced the question to a model case.
- Compute the Bergman kernel of the model case.

Remark

- The above theorem can be viewed as a refined solution of the Levi problem proved by Oka.
- One may also ask for L^p-estimate of $\bar{\partial}$, closely related to p-Bergman kernels.

Skoda's L^2-division theorem

Theorem (Skoda's L^2-division theorem, '72)

Let Ω be a domain spread over $\mathbb{C}^n$ which is Stein. Let ψ be a psh function on Ω, $g_1, \cdots, g_p$ be holomorphic functions on Ω, $\alpha > 1$, $q = \min\{n, p-1\}$, and f be a holomorphic function on Ω. Assume that

$$\int_\Omega \frac{|f|^2 e^{-\psi}}{(\sum_{j=1}^p |g_j|^2)^{\alpha q+1}} \leq +\infty.$$

Then there exists holomorphic functions $h_1, \cdots, h_p$ on Ω with $f = \sum_{j=1}^p h_j g_j$ on Ω such that

$$\int_\Omega \frac{|h_k|^2 e^{-\psi}}{(\sum_{j=1}^p |g_j|^2)^{\alpha q}} \leq \int_\Omega \frac{|f|^2 e^{-\psi}}{(\sum_{j=1}^p |g_j|^2)^{\alpha q+1}}$$

for $1 \leq k \leq p$.

Remarks

Reference:
H. Skoda. Applications des techniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids, Ann. Scient. Ec. Norm. Sup.4e Série, 5, 545–579, 1972.

It is a effective variant of Oka's division theorem. See reference below:

K. Oka. Sur les fonctions analytiques de plusieurs variables. VII. Sur quelques notions arithmétiques. Bull. Soc. Math. France 78,1–27(1950).

Ohsawa-Takegoshi extension theorem

Theorem (Ohsawa-Takegoshi extension theorem, '87)

Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and $V \subset \Omega$ be a smooth subvariety of Ω. Let $\phi \subset Psh(\Omega)$ such that $\phi|_V \not\equiv -\infty$. Then for any holomorphic function f on V such that $\int_V |f|^2 e^{-\phi} < +\infty$, there is a holomorphic function F on Ω, such that

$$\int_\Omega |F|^2 e^{-\phi} \le C \int_V |f|^2 e^{-\phi|_V},$$

where C is a uniform constant only depending on the diameter of Ω and the codimension of V in Ω. (In particular, C is independent of f and ϕ.)

T. Ohsawa and K. Takegoshi. On the extension of L^2 holomorphic functions, Math. Z., 195, 411–421, 1987.

Motivation of OT-theorem

T. Ohsawa. A survey on the L^2 extension theorems, J. Geom. Ana. 2020.

extension theorem for holomorphic functions with L^2 growth conditions. Although the author's naïve motivation in [93] was to extend Theorem 0.1 to the case of weakly pseudoconvex boundary points, the result has been applied to solve many questions in complex analysis and geometry, as well as [58] and [112]. Moreover, by recent works

Theorem 0.1 here is the theorem of Hörmander to solve Bergman's question.

Ohsawa-Takegoshi L^2-extension with optimal constant

Theorem (A simplified version of Guan-Zhou)

Let $B \subset \mathbb{C}^r$ be the unit ball and let $p : X \to B$ be a projective family, and a proper submersion. Let $L \to X$ be a holomorphic line bundle on X with a singular Hermitian metric h whose curvature current is positive. Let $u \in H^0(X, K_{X_0} \otimes L \otimes \mathcal{I}(h)|_{X_0})$, such that $\int_{X_0} u \wedge \bar{u} h < \infty$. Then there exists $s \in H^0(X, K_X \otimes L \otimes \mathcal{I}(h))$ such that $S|_{X_0} = u \wedge dt$ and

$$\int_X s \wedge \bar{s} h \leq \mu(B) \int_{X_0} u \wedge \bar{u} h.$$

where $t = (t_1, \cdots, t_r)$ is the standard coordinate on B and $dt = dt_1 \wedge \cdots \wedge dt_r$, and $\mu(B)$ is the volume of B with respect to the Lebesgue measure on B.

Remark

- Domain case was first proved by Błocki, refinement of the ordinary differential equation technique of Zhu-Guan-Zhou.
- Extended to weakly pseudoconvex Kähler case by Zhou-Zhu, Cao.
- $L^{\frac{2}{m}}$-version was first proved by Berndtsson-Paun, L^p by Guan-Zhou.

Reference:

Z. Błocki, Suita conjecture and the Ohsawa-Takegoshi extension theorem. Invent. Math., 193(1):149–158,2013.

Q. Guan and X. Zhou, A solution of an L^2 extension problem with an optimal estimate and applications. Ann. of Math. (2),181(3):1139–1208, 2015.

X. Zhou and L. Zhu, An optimal L^2-extension theorem on weakly pseudoconvex Kähler manifolds, J. of Differential Geom., 110(1):135–186, 2018.

X, Zhou and L. Zhu, Siu's lemma, optimal L^2 extension, and applications to twisted pluricanonical sheaves, Math. Ann.,377(2020),675–722.

Motivation: Suita conjecture

Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω. Let B_Ω be the Bergman kernel metric for K_Ω. Let $c_\beta(z)$ be the logarithmic capacity which is locally defined by $c_\beta(z_0) = \exp\lim_{z\to z_0}(G_\Omega(z, z_0) - \log|w(z)|)$ on Ω, where w is a local holomorphic coordinate near z_0.

Conjecture (Suita conjecture)

On any open Riemann surface Ω, $c_\beta^2(z_0) \le \pi B_\Omega(z_0)$, with equality if and only if Ω is conformally equivalent to the unit disk less a (possible) closed set of inner capacity zero.

It is proved that OT-extension implies that $c_\beta^2(z_0) \le C\pi B_\Omega(z_0)$, where C is the same constant as in OT extension theorem (e.g. see paper: [L.Zhu, Q. Guan and X. Zhou, On the Ohsawa-Takegoshi L^2 extension theorem and the Bochner-Kodaira identity with non-smooth twist factor, J. Math. Pures Appl. 97 (2012),579–601.]

Contents

- Vanishing theorems
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- Siu's invariance of plurigenera
- Siu's analytic proof of finite generation of pluricanonical ring

Nadel vanishing theorem: Hörmander's L^2-estimate

Theorem (Nadel '89)

Let (X, ω) be a compact Kähler manifold, and let F be a holomorphic line bundle over X equipped with a singular Hermitian metric h of weight φ. Assume that $i\Theta_{F,h} \geq \varepsilon\omega$ for some continuous positive function ε on X. Then

$$H^q(X, \mathcal{O}(K_X + F) \otimes \mathcal{I}(h)) = 0 \quad \text{for} \quad \text{all} \quad q \geq 1.$$

A.M.Nadel. Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature, Proc. Nat. Acad. Sci. U.S.A., 86,7299-7300, 1989. and Ann. of Math. 132,549–596,1990.

Remark

Demailly extended this result to Kähler weakly pseudoconvex manifold.

Remarks

- The proof of coherence of multiplier ideal sheaf $\mathcal{I}(\varphi)$ depends on Hörmander's L^2-estimate.
- Both the proof of acyclic resolution of coherent analytic sheaves $(\mathcal{O}(K_X + F) \otimes (h))$ and vanishing of cohomology of L^2-sheaves depend on Hörmander's L^2-estimate.
- When the metric h is smooth, it recovers Nakano vanishing theorem.
- It can be used to prove Kawamata-Viehweg vanishing theorem, however, it is more than Kawamata-Viehweg Vanishing theorem, and plays important role to produce sections in algebraic geometry.
- There is also a modified Hörmander type L^2-estimate for $\bar{\partial}$ with error term, which is also used to get vanishing theorems for singular or degenerate metrics, e.g. Demailly-Peternell(-Schneider), Cao, Matsumura...

Yau's problem: Hörmander's L^2-estimate

Question

Whether or not the Kähler-Einstein metric on M can be the limit of a sequence of Bergmann mertics induced by pluricanonical line bundles K_M^m?

Theorem (Tian '90)

Let M be an algebraic manifold with a polarized L and let g be a polarized Kähler metric. Then g can be the limit of a sequence of Bergman mertics induced by pluricanonical line bundles K_M^m in $\mathcal{C}^2$.

- A Key step in Tian's proof is to use Hörmander's L^2-estimate to construct peak sections.
- Inspired others to study the expansion of Bergman kernels which is very important in the study of Yau-Tian-Donaldson's conjecture.

Remarks

- In recent study of Yau-Tian-Donaldson's conjecture, Hörmander's L^2-estimate is combined with Gromov-Hausdorff theory to get a partial C^0-estimate for Bergman kernels.

Reference:

G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, J. Differential Geometry, 32 (1990),99–130.

Demailly's regularization of psh functions: Hörmander's L^2-estimate+OT L^2-extension (Model case)

Theorem (Demailly '92)

Let φ be a psh function on a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$. For every $m \in \mathbb{N}^+$, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2m\varphi} d\lambda < \infty$ and let $\varphi_m = \frac{1}{2m} \log \sum |\sigma_l|^2$ where (σ) is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$. Then there are constants $C_1, C_2 > 0$ independent of m such that

a. $\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{|\xi - z| < r} \varphi(\xi) + \frac{1}{m} \log \frac{C_2}{r^n}$ *for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$ and*

b. $\nu(\varphi, z) - \frac{n}{m} \leq \nu(\varphi_m, z) \leq \nu(\varphi, z)$ *for every $z \in \Omega$.*

J.-P. Demailly, Regularization of closed positive currents and intersection theory, J. Alg. Geom., 1 (1992), 361–409.

Compared with Bremerman's result

Theorem (Bremerman '56)

Any psh function u on a Stein manifold D can be represented in the form

$$u(z) = (\limsup \alpha_j \ln |f_j|)^*, \quad \alpha_j \geq 0, \quad f_j \in \mathcal{O}(D).$$

H.J. Bremermann, On the conjecture of the equivalence of the plurisubharmonic functions and the Hartogs functions, Math. Ann. 131, No.1, 76–86, 1956.

- For planar domains, proved by Lelong based on classical potential theory and rather long.
- Demailly-pointwise limit, Bremermann-upper semicontinuous regularization.

Siu's semicontinuity of Lelong number upper level sets

Theorem (Siu '74)

Let φ be a psh function on a complex manifold. Then for every $c > 0$, the Lelong number upperlevel set $E_c(\varphi) = \{z \in X, \nu(\varphi, z) \geq c\}$ is an analytic subset in X.

Proof.

We may assume $X = \Omega$. Note that $E_c(\varphi) = \cap_{m \geq m_0} E_{c-n/m}(\varphi_m)$. Since φ_m is with analytic singularities, then $E_{c-n/m}(\varphi_m)$ is an analytic subvariety, from strong Noether property, $E_c(\varphi)$ is an analytic subvariety. □

Remark

Applies to prove the same conclusion for any closed positive (p, p) currents by combining a result of Skoda.

On Compact Hermitian manifold

Theorem (Demailly)

Let T be a closed almost positive $(1,1)$-current on a compact complex manifold X, and fix a Hermitian form ω. Suppose that $T \geq \gamma$ for some smooth real $(1,1)$-form γ on X. Then

1. *There exists a sequence of smooth forms $\theta_k \in [T]$ which converges weakly to T, and such that $\theta_k \geq \gamma - C\lambda_k\omega$ where $C > 0$ is a constant depending on teh curvature of (T_X, ω) only, and λ_k is a decreasing sequence of continuous functions such that $\lambda_k \to \nu(T,x)$ for evey $x \in X$.*
2. *There is a sequence T_k of currents with analytic singularities in $[T]$ which converges weakly to T, such that $T_k \geq \gamma - \varepsilon_k\omega$ for some sequence $\varepsilon_k > 0$ decreasing to 0, and such that $\nu(T_k,x) \to \nu(T,x)$ uniformly w.r.t. $x \in X$.*

Equisingular approximations of quasi-psh functions

Theorem (Demailly-...continued)

3. One can write $\varphi = \lim_{\nu\to\infty} \varphi_\nu$ where

- *φ_ν is smooth in the complement $X \setminus Z_\nu$ of an analytic set $Z_\nu \subset X$;*
- *$\{\varphi_\nu\}$ is a decreasing sequence, and $Z_\nu \subset Z_{\nu+1}\}$ for all ν;*
- *$\mathcal{I}(\varphi_\nu) = \mathcal{I}(\varphi)$ for all ν;*
- *$T_\nu = \alpha + i\partial\bar{\partial}\varphi_\nu$ satisfies $T_\nu \geq \gamma - \varepsilon_\nu\omega$ with $\varepsilon_\nu \to 0$.*

Remark

From model case to compact manifold case, one should patch local data, where Hörmander's estimate was used to estimate the difference.

J.-P. Demailly, T. Peternell and M.Schneider, Pseudoeffective line bundles on compact Kähler manifolds. Internat. J. Math. 12(2001), 689–741.

Applications of Demailly's approximation theorems

- Boucksom: Divisorial Zariski decomposition and Mobile intersection theory
- BDPP: Dual of pseudoeffective cone on projective manifolds ($\mathcal{E}_{NS} = (\overline{ME^s(X)})^*$)
- DPS: Hard Lefschetz theorem for singular metrics
- BEGZ: Non-pluripolar product
- Cao: Kawamata-Viehweg-Nadel type vanishing theorems
- Matsumura: vanishing theorems and injective theorems

Fujita's conjecture: Riemann-Roch+Nadel vanishing

Conjecture (Fujita's conjecture)

Let X be a projective manifold of dimension n. Let $L \to X$ be a ample line bundle. Then

- $(n+1)L + K_X$ *is free,*
- $(n+2)L + K_X$ *is very ample.*

1. $n = 1$, proved by Reider '88. Freeness part for $n = 3$, proved by Ein-Lazarsfeld '93.
2. Demailly '93, proved the very ampleness of $12n^nL + K_X$ and the generation of r-jets by global sections of $6(n+r)^nL + 2K_X$.
3. Siu '94, proved the generation of r-jets by global section of $mL + 2K_X$ for $m \geq 2(n + 2 + n\binom{3n+2r-1}{n})$, Demailly '94 improved to $2 + \binom{3n+2r-1}{n}$.

- Angehrn-Siu '95: $mL + K_X$ globally generated by global section for $m \geq \frac{1}{2}(n^2 + n + 2)$. Global sections of $mL + K_X$ seperate any set of r-distinct points for $m \geq \frac{1}{2}(n^2 + 2rn - n + 2)$.
- Kollar '93, Ein-Küchle-Lazarsfeld '94, Ein-Lazarsfeld-Nakamaye '94 also make progress to Fujita's conjecture.
- Siu '96: $mL + K_X$ is very ample for $m \geq 2(n + 2 + n\binom{3n+1}{n})$.
- Zhu-Ye '15: a proof of freeness part in dimension 5.
- ....

Berndtsson's psh variation of relative Bergman kernels: Hörmander's L^2-estimate

Theorem (Berndtsson '06)

Let $\Omega \subset \mathbb{C}_t^m \times \mathbb{C}_z^m$ be a bounded pseudoconvex domain, and $\psi(t,z) \in Psh(\Omega)$. Let $K_t(z) := B(\Omega_t, \psi(t,\cdot))$ the Bergman kernel of Ω_t w.r.t the weight $\psi(t,\cdot)$, then $\log K_t(z)$ is psh on Ω if it is not identically $-\infty$.

- Berndtsson using Hörmander's L^2-estimate.
- Guan-Zhou using optimal L^2-extension for projective family.

B. Berndtsson. Subharmonicity properties of the Bergman kernel and some other functions assocaited t pseudoconvex domains. Ann. Inst. Fourier (Grenoble) 56(6),1633-1662, 2006.

All in all we therefore have that

$$\text{(2.3)} \qquad \frac{\partial^2 \Phi}{\partial t \partial \bar{t}} = \int_V |\bar{\partial}_t K_t|^2 \mathrm{e}^{-\phi^t} + \int_V \phi_{t\bar{t}} |K_t|^2 \mathrm{e}^{-\phi^t} - \int_V |u|^2 \mathrm{e}^{-\phi^t}.$$

To estimate the last term we note that u solves the $\bar{\partial}$-equation

$$\bar{\partial} u := f = \bar{\partial}\, \partial_t^{\phi} K_t = K_t \bar{\partial}\, \frac{\partial \phi}{\partial t},$$

(the last equation follows from a commutation rule similar to (2.2) since K_t is holomorphic). Moreover, u is the minimal solution to this equation, since u is orthogonal to the space of holomorphic functions. By Hörmander's theorem (see [8] for an appropriate formulation) we therefore get that

$$\int_V |u|^2 \mathrm{e}^{-\phi^t} \leqslant \int_V \sum (\phi^t)^{j\bar{k}} f_j \bar{f}_k \mathrm{e}^{-\phi^t},$$

where $(\phi^t)^{j\bar{k}}$ is the inverse of the complex Hessian of ϕ^t. Inserting this into (2.3) and discarding the first (nonnegative) term we have

$$\frac{\partial^2 \Phi}{\partial t \partial \bar{t}} \geqslant \int_V |K_t|^2 D \mathrm{e}^{-\phi^t},$$

where

$$D = \phi_{t\bar{t}} - \sum (\phi_{z_j \bar{z}_k})^{-1} \phi_{t\bar{z}_j} \overline{\phi_{t\bar{z}_k}}.$$

Guan-Zhou's method

Let Δ_r be the unit disc with center (z,t_0) and radius r on the line $\{t|(z,t)\}$. In Theorem 2.2, let $\Psi=\log|t|^2$ and $c_A\equiv 1$, where $A=2\log r$. We obtain a holomorphic section $\tilde{u}$ on $p^{-1}(p(\Delta_r))$ such that

$$\int_{M_{t_0}}\{u_{t_0},u_{t_0}\}_h^2\geq\frac{1}{\pi r^2}\int_{\Delta_r}\int_{M_t}\left\{\frac{\tilde{u}}{dt}|_{M_t},\frac{\tilde{u}}{dt}|_{M_t}\right\}_h d\lambda_{\Delta_r}(t), \tag{3.1}$$

where $\tilde{u}=\tilde{g}(z,t)dz\wedge dt\otimes e$ on (z,t) and $\tilde{g}(z,t_0)=g(z)$.

Using the extremal property of the Bergman kernel, we have

$$B_t(z)\geq\frac{|\tilde{g}(z,t)|^2}{\int_{M_t}\frac{1}{2^n}\{\frac{\tilde{u}}{dt}|M_t,\frac{\tilde{u}}{dt}|M_t\}_h},$$

for any $(z,t)\in\Delta_r$, if $\int_{M_t}\{\frac{\tilde{u}}{dt}|_{M_t},\frac{\tilde{u}}{dt}|_{M_t}\}_h\neq 0$.

Note that the Lebesgue measure of $\{t|\int_{M_t}\{\frac{\tilde{u}}{dt}|_{M_t},\frac{\tilde{u}}{dt}|_{M_t}\}_h=0\}$ is zero. Using convexity of function $y=e^x$ and inequality (3.1), we have

$$e^{2\log|g(z)|-\log B_{t_0}(z)}=\frac{|g(z)|^2}{B_{t_0}(z)}\geq e^{\frac{1}{\pi r^2}\int_{\Delta_r}(2\log|\tilde{g}(z,t)|-\log B_t(z))d\lambda_{\Delta_1}(t)}. \tag{3.2}$$

Positivity of direct images

Theorem (Berndtsson '09)

Let $\pi : X \to Y$ be a proper submersion and X is a Kähler manifold. Let $L \to X$ be a semipositive (resp. strict positive) holomorphic line bundle, then the direct image $p_*(K_{X/Y} + L)$ is Nakano semi-positive (resp. strict Nakano positive) .

Proved by using standard Kähler geometry technique, not much clear relation with Hörmander's L^2.

> We want to add one remark on the relation between the proof of Theorem 1.2 in this section and the proof of Theorem 1.1 in Section 3. The proof in Section 3 is easily adapted to the case of a trivial fibration (so that X is a global product). It may then seem that the proof here is quite different since it does not use the Hörmander-Kodaira L^2-estimates at all. The two proofs are however really quite similar, the difference being that in this section we basically reprove the special case of the L^2-estimates that we need as we go along.

B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations. Ann. of Math. 169 (2009),531–560.

GuanZhou's solution of SOC: OT extension

Theorem (Guan-Zhou '15)

Let φ be a psh function on a complex manifold X. Then

$$\mathcal{I}_+(\varphi) := \cup_{\varepsilon>0}\mathcal{I}((1+\varepsilon)\varphi) = \mathcal{I}(\varphi).$$

theorem.

In the reviewer's opinion the proofs of both the openness and the strong openness conjectures are among the greatest achievements "in the intersection" of complex analysis and algebraic geometry in recent years.

Reviewed by Żywomir Dinew

Q. Guan and X. Zhou. A proof of Demailly's strong openness conjecture, Ann. of Math., 182(2015), 1–12.

Siu's invariance of plurigenera: OT L^2 extension + Skoda's L^2 division

Theorem (Siu '98, '02)

Let $X \to S$ be a proper holomorphic submersion and a projective family on an irreducible base S. Then the plurigenus $p_m(X_t) = h^0(X_t, mK_{X_t})$ is independent of t for all $m \geq 0$.

No algebraic proof in the case of varieties of non-negative Kodaira dimension which are not of general type!!!

Conjecture (Siu's conjecture)

The above conjecture holds for Kähler families.

Y.-T. Siu, Invariance of plurigenera. Invent. Math. 134(1998),661–673.
Y.-T. Siu, Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type. In: Complex

Siu's analytic proof of finite generation: a key ingredient is Skoda's L^2-division theorem

Theorem (Siu '06)

Suppose the stable vanishing orders are precisely achieved at every point of X for some $m_0 \in \mathbb{N}$. Denote $(m_0!)$ by m_1. Then the canonical ring

$$\oplus_{m=1}^{\infty}\Gamma(X, mK_X)$$

is generated by

$$\oplus_{m=1}^{(n+2)m_1}\Gamma(X, mK_X).$$

Y.-T. Siu, A general non-vanishing theorem and an analytic proof of the finite generation of the canonical ring, arXiv:0610740
Y.-T. Siu, Finite generation of canonical ring by analytic method. Sci. China Ser. A 51(2008),481–502.

Define

$$\Phi = \sum_{m=1}^{\infty} \varepsilon_m \sum_{j=1}^{q_m} |s_j^{(m)}|^{\frac{2}{m}}, \Phi_{m_0} := \sum_{m=1}^{m_0} \varepsilon_m \sum_{j=1}^{q_m} |s_j^{(m)}|^{\frac{2}{m}}$$

with $s_1^{(m)}, \cdots, s_{q_m}^{(m)} \in \Gamma(X, mK_X)$ form a basis over $\mathbb{C}$ and $\varepsilon_m \to 0$ very fast to make Φ convergent. We say that the stable vanishing order of Φ is precisely achieved at p, if for some m_0 the function, there exists some open neighborhood U of p in X and some positive number C such that

$$\frac{1}{C}\Phi_{m_0} \leq \Phi \leq C\Phi_{m_0}$$

on U.

Motivation

Table: Converse of L^2-theory

Converse Hörmander's L^2-estimate	Converse OT L^2-extension
Multiple coarse L^p-estimate property	Multiple coarse L^p-extension property
Optimal L^p-estimate property	Optimal L^p-extension property

Note that if ϕ is psh, the above four properties hold for ϕ with $p = 2$, by Hörmander's L^2-estimate and OT extension theorem, the so called Converse of L^2-theory is asking the following

Question

Give an upper semi-continuous function ϕ on Ω, if ϕ satisfies one of the above properties, if ϕ psh?

Main results

Theorem

(a) *If ϕ is $\mathcal{C}^2$, and satisfies the optimal L^2-estimate property, then ϕ is psh. [DNW, arXiv: 1910.06518]*

(b) *If ϕ is continuous, and satisfies the multiple coarse L^p-estimate property, then ϕ is psh. [DNW, arXiv: 1910.06518]*

(c) *If ϕ is upper semi-continuous, and satisfies the optimal L^p-extension property, then ϕ is psh. [DNW, arXiv: 1910.06518]*

(d) *If ϕ is upper semi-continuous, and satisfies the multiple coarse L^p-extension property, then ϕ is psh. [DWZZ,arXiv: 1809.10371], and a new proof in [DNW, arXiv: 1910.06518]*

The above theorems show that the four properties are all equivalent to the plurisubharmonicity of ϕ, which gives us a satisfied picture.

Remarks

- (b) solves a problem raised in Hosono-Inayama's paper, on removing the locally uniformly Hölder continuity assumption.
- New ideas in the proof of (a): connecting $i\partial\bar{\partial}\phi$ with the optimal L^2-estimate property via a Bochner type identity, and then using a localization technique to produce a contradiction if ϕ is assume to be not psh.
- (c) and (d) was proved by using Guan-Zhou's method and an analytic lemma.

Positivities of holomorphic vector bundles

We also define similar four properties for hermitian holomorphic vector bundles (E, h), and we proved that

Theorem (DNWZ, arXiv:2001.01762)

- *Optimal L^2-estimate property implies Nakano semi-positivity.*
- *Multiple coarse L^p-estimate property for $p > 1$ implies Griffiths semi-positivity.*
- *Optimal L^p-extension property implies Griffiths semi-positivity.*
- *Multiple coarse L^p-extension property implies Griffiths semi-positivity.*

Positivities of direct image bundles: revisited

Let $p : X \to Y$ be a proper holomorphic submersion of Kähler manifolds, and (E, h) be a hermitian holomorphic vector bundle on X, with Nakano semi-positive curvature. Let $p_*(K_{X/Y} \otimes E)$ be the direct image bundle, which endows a natural hermitian metric by fiberwise L^2-integration. We denote it by $(F, \|\cdot\|)$.

Theorem (DNWZ, arXiv:2001.01762)

The direct image bundle $(F, \|\cdot\|)$ satisfies the optimal L^2-estimate property, the multiple coarse L^2-estimate property, the optimal L^2-extension property, the multiple coarse L^2-extension property. In particular, $(F, \|\cdot\|)$ is Nakano semi-positive.

Zhou-Zhu also proved the Griffiths positivity based on Guan-Zhou's method and their new optimal L^2-extension theorem on weakly pseudoconvex Kähler manifolds.

Thanks for your attention!